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Optimal Management of Defined Contribution Pension Funds within a Model Uncertainty Framework

Gerhard-Wilhelm Weber *

Faculty of Engineering Management, Chair of Marketing and Economic Engineering,
Poznan University of Technology, Poland;
Institute of Applied Mathematics, METU, Ankara, Turkey

EURO THE ASSOCIATION OF EUROPEAN OPERATIONAL RESEARCH SOCIETIES

Ioannis Baltas, Athanasios Yannacopoulos, Marek Szcepanski, Krzysztof Kolodziejczyk, Lukasz Dopierala

* Faculty of Economics, Management and Law, University of Siegen, Germany
School of Science, Information Technology and Engineering, Federation University, Ballarat, Australia
Center for Research on Optimization and Control, University of Aveiro, Portugal
Universiti Teknologi Malaysia, Skudai, Malaysia
Universitas Sumatera Utara, Medan, Indonesia
Mazandaran University of Science and Technology, Babol, Iran
Vidyasagar University, Midnapore, India
Georgian International Academy of Sciences
Dept. Industrial & Systems Engineering, College of Engineering, Istinye University, Istanbul, Turkey
Advisor to EURO Conferences

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Gerhard-Wilhelm Weber[1], **Yiannis Baltas**[2,4] & **Athanasios N. Yannacopoulos**[3,4]

[1]Faculty of Engineering Management
Chair of Marketing and Economic Engineering
Poznan University of Technology, Poland;
Institute of Applied Mathematics, METU, Ankara, Turkey

[2]Department of Financial and Management Engineering
University of the Aegean, Greece

[3]Department of Statistics
Athens University of Economics and Business, Greece

[4]Stochastic Modeling and Applications Laboratory
Athens University of Economics and Business, Greece

In Collaboration with:

Marek Szczepánski

Head of Chair of Economic Sciences
Faculty of Engineering Management
Poznan University of Technology, Poland

Krzysztof Kolodziejczyk

Faculty of Management Engineering
Department of Economic Sciences
Poznan University of Technology, Poland

Lukasz Dopierala

Institute of Foreign Trade
Department of International Financial Markets
University of Gdansk, Poland

It's all about uncertainty

Uncertainty is an uncomfortable position.
But certainty is an absurd one.

Voltaire

Life's most precious gift is uncertainty.

Yoshida Kenko

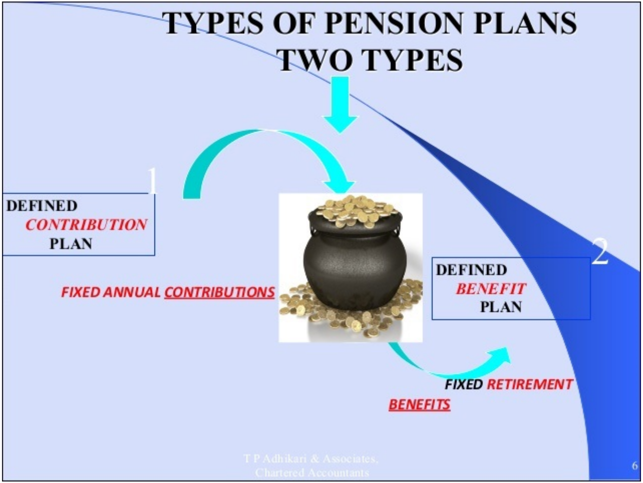
Uncertainty that comes from knowledge is different
from uncertainty that comes from ignorance.

Isaac Asimov

Optimal management of pension funds

- Saving for retirement - like any form of long-term savings - involves various risks.
- The problem of optimal management of assets accumulated in pension funds under risk and uncertainty is a matter of fundamental importance - both in the theoretical and practical dimension.
- Essentially, a pension fund scheme constitutes an independent legal entity that represents accumulated wealth stemming from pooled contributions of its members.
- This wealth is to be invested over a long period of time
- (usually from 20 to 40 years) in order to provide its members with retirement benefits (in the form of periodic pension payments or a one-off payment).

Optimal management of pension funds



Optimal management of pension funds

- In general, as already mentioned, there are two completely different methods to design a pension fund scheme: (i) the defined benefit plan (DB), and (ii) the defined contribution plan (DC).
- **According to a DC plan:**
  - every member of the fund **contributes a fixed proportion** of his/her income (before retirement), which are collected in an individual investment account and the benefits to be received (after retirement) consist of a fraction of the true fund value.
  - Thus, they solely depend on the investment performance of the fund portfolio during its lifetime.
- **According to a DB plan:**
  - the benefits are initially fixed while the contributions are dynamically adjusted in order to keep the fund in balance.
  - In other words, according to the DB plan it is the contributions which are random.

Optimal management of pension funds

- Since the success of a DC plan crucially depends on the effective investment of the available funds due to contributions, the optimal management of the fund reduces to the problem of **optimal portfolio selection** from an available collection of financial assets.
- Pay attention to the stochastic nature of the problem at hand!

Stochastic control techniques & pension funds

- Di Giacinto, M., Federico, S. & Gozzi, F. Pension Funds with a Minimum Guarantee: A Stochastic Control Approach. Finance and Stochastics, **15**, 297–342 (2011).
- Battocchio, P & Menoncin, F.: Optimal portfolio strategies with stochastic wage income and inflation. the case of a defined contribution pension fund. CeRP Working Papers **19** (2002).
- Boulier, J.F., Huang & S., Taillard, G.: Optimal management under stochastic interest rates: the case of a protected defined contribution pension fund. Insurance: Mathematics & Economics **28**, 173-189 (2001).
- Deelstra, G.: Optimal design of the guarantee for defined contribution funds. Journal of Economic Dynamics and Control **28**, 2239-2260 (2004).
- Guan, G., Liang, Z.: Optimal management of DC pension plan in a stochastic interest rate and stochastic volatility framework. Insurance: Mathematics and Economics **57**, 58-66 (2004).

The model I: Financial Market

Suppose that we have a financial market on the fixed time horizon $[0, T]$ with $T > 0$ and three investment possibilities:

Asset 1

A zero coupon bond with maturity T and dynamics described by

$$\begin{aligned} \frac{dP(t,T)}{P(t,T)} &= (r + \alpha\theta)dt + \alpha dW_1(t), \\ P(0,T) &> 0, \end{aligned} \tag{1}$$

where

- $P(t, T)$ denotes the price of the bond at time $t \in [0, T]$.
- $r > 0$ and $\alpha > 0$ stand, respectively, for the interest rate and the volatility of bond prices.
- $\alpha\theta$ (for some $\theta > 0$) stands for the excess return on the bond.
- W_1 is a Brownian motion.

The model I: Financial Market

Asset 2

Another risky asset (e.g., a financial index or stock) which evolves according to the stochastic differential equation

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW_2(t), \qquad (2)$$
$$S(0) = S_0 > 0,$$

where

- $S(t)$ denotes the price of the index at time $t \in [0, T]$.
- $\mu > r > 0$ stands for the appreciation rate of the stock prices.
- $\sigma > 0$ stands for the volatility of the stock prices.
- W_2 is another Brownian motion.

The model I: Financial Market

Asset 3

A risk free asset (bank account) with unit price $B(t)$ at time $t \in [0, T]$ and dynamics described by the ordinary differential equation

$$\begin{aligned} dB(t) &= rB(t)dt, \\ B(0) &= 1. \end{aligned} \tag{3}$$

Key points

- Market parameters are assumed to be constants for simplicity.
- Extension with time varying parameters is possible (but painful).
- The market is complete, as we have two noises and two traded assets.
- Brownian motions are assumed orthogonal for algebraic simplicity.

The model II: Salaries

- Salaries are in general stochastic (affected by macroeconomic and microeconomic factors).
- We consider the stochastic process $(L(t);\, t \geqslant 0)$ that denotes the average salary at time $t \in [0, T]$ and is assumed to obey the probabilistic law:

$$\begin{aligned} \frac{dL(t)}{L(t)} &= \mu_L(r)dt + k_2 dW_1(t) + k_3 dW_2(t), \\ L(0) &= l_0 > 0, \end{aligned} \tag{4}$$

<u>where</u>

- $l_0 \in \mathbb{R}_+$ denotes the initial average salary level.
- $\mu_L(r)$ is the expected instantaneous growth rate of the average salaries.
- $k_2, k_3 \in \mathbb{R}$ are scaling factors that describe the effect that bond market and stock market have on the evolution of the average salary.

The Pension Fund Setting

- We consider a DC pension fund scheme.
- Employees that become part of the pension fund have to pay contributions.
- Contributions are assumed to be paid continuously at rate q.
- Here $qL(t)$ denotes the aggregate contributions up to time $t \in [0, T]$.
- We also envision a fund manager, who, at time $t = 0$, is endowed with some initial wealth $x > 0$.
- $\pi(t)$: proportion of fund's wealth invested in the stock.
- $b(t)$: proportion of fund's wealth invested in the bond.
- What remains is invested in the remaining asset (bank account).

Stochastic Differential Equation of Fund's wealth

- The fund's wealth process corresponding to the strategy $(\pi(t), b(t))$, is denoted as $X(t)$ and is defined as the solution of the following linear stochastic differential equation:

$$dX(t) = \pi(t)X(t)\frac{dS(t)}{S(t)} + b(t)X(t)\frac{dP(t,T)}{P(t,T)} + (1 - \pi(t) - b(t))\,X(t)\frac{dB(t)}{B(t)} + qL(t)dt.$$

- Therefore, in view of (1)-(4):

$$\begin{aligned} dX(t) &= \left([r + \pi(t)(\mu - r) + \alpha\theta b(t)]\,X(t) + qL(t)\right)dt \\ &\quad + \sigma\pi(t)X(t)dW_1(t) + \alpha b(t)X(t)dW_2(t), \qquad (5) \\ X(0) &= x > 0. \end{aligned}$$

The Original Problem

- The fund manager chooses the control processes so as to maximize some certain goal, e.g., the expected utility from her terminal relative wealth:

$$\sup_{(\pi,b)\in\mathcal{A}^{\mathbb{F}}} \mathbb{E}\left[U\left(Y(T)\right)\right], \;\; Y(T) = \frac{X(T)}{L(T)},$$

  subject to the state process

$$\begin{aligned} dX(t) &= \left(\left[r + \pi(t)(\mu - r) + \alpha\theta b(t)\right] X(t) + qL(t)\right) dt \\ &\qquad + \sigma\pi(t)X(t)dW_1(t) + \alpha b(t)X(t)dW_2(t), \\ \frac{dL(t)}{L(t)} &= \mu_L(r)dt + k_2 dW_1(t) + k_3 dW_2(t). \end{aligned}$$

  with initial conditions $X(0) = x$, $L(0) = l_0 > 0$.
- A standard way to proceed is by employing the techniques with stochastic optimal control.

Stochastic Optimal Control

<u>Main Idea</u>

- The underlying system is represented by a controlled stochastic process.
- The decision maker chooses the control process to drive the system to the desired state.
- Many applications in a variety of fields:
  - Mathematical Finance
  - Insurance
  - Risk Management, etc.

<u>Main Assumption</u>

- The decision maker blindly trusts the model he faces.
- The exact probability law of the stochastic risk factors in the underlying model, is precisely known.

Robust Control Theory

- An important part of stochastic control.
- In some sense it is the most realistic version of control theory.

Main Idea

- We wish to control a system but we do not know the exact law of evolution of the state process.
- What we have is a family of laws (scenarios), and we want to control the worst possible scenario.
- The best policy for the worst scenario is our robust control.

Stochastic Control vs Robust Control

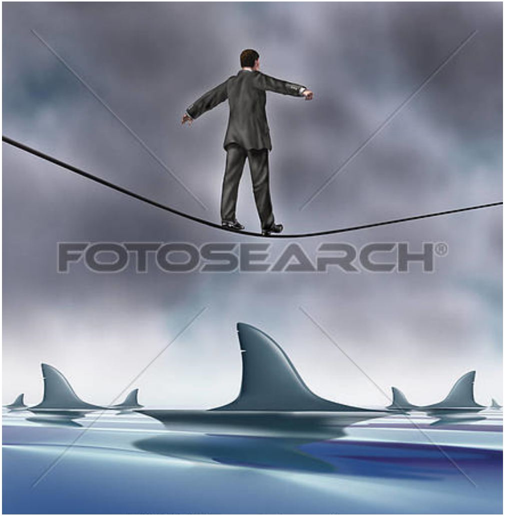


Figure: Stochastic Optimal Control Theory

Stochastic Control vs Robust Control

Figure: Robust Optimal Control Theory

Introduction to Robust Control Theory

Robust control theory is a mixture of two things:

- Stochastic control theory.
- Model selection techniques.

Main Philosophy:

Solve an optimal control problem under the worst possible scenario.
$\Longrightarrow$ Using the model that may provide the worst case for the problem at hand.

In Mathematical terms:

Model $\sim$ Probability Measure

Model Uncertainty Aspects

- Uncertainty concerning the "true" statistical distribution of the state of the system.
- We assume that the controller is uncertain as to the true nature of the stochastic processes W_1 and W_2 in the sense that the exact law of W_1 and W_2 is not known.
- There exists a "true" probability measure related to the true law of the process W_1 and W_2, the controller is unaware of and a probability measure $\mathbb{Q}$, which is his/her idea of what the exact law of W_1 and W_2 looks like.
- As the controller is uncertain about the validity of $\mathbb{Q}$ as a proper description of the futures states of the world, she seeks to make her decision robust.

Model Uncertainty Aspects

- She adopts a "cautionary" approach that of seeking to maximize the worst possible scenario concerning the true description of the noise term. This is quantified as:

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\Big[U(Y(T))\Big],$$

- As a result, the manager faces the robust control problem

$$\sup_{\pi\in\mathcal{A}^{\mathbb{F}}} \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\Big[U(Y(T))\Big],$$

The Class of Measures $\mathcal{Q}$

Definition (The set $\mathcal{Q}$)

The set of acceptable probability measures $\mathcal{Q}$ for the agent is a set enjoying the following two properties:

(i) Considering the stochastic process $W := (W_1, W_2)$ under the reference probability measure $\mathbb{P}$ and under the probability measure $\mathbb{Q}$ results to a change of drift to the Brownian motion W.

(ii) There is a maximum allowed deviation of the managers measure $\mathbb{Q}$ from the reference measure $\mathbb{P}$. In other words, the manager is not allowed to freely choose between various probability models as every departure will be penalized by an appropriately defined penalty function, a special case of which is the Kullback-Leibler relative entropy $\mathcal{H}(\mathbb{P}|\mathbb{Q})$.

Change of measure - Girsanov

Theorem

Assume that $u, \lambda \in \mathcal{Y} \subset \mathbb{R}^2$ *satisfy the condition*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \left(u^2(s) + \lambda^2(s)\right)\right)ds\right] < \infty.$$

Then, the stochastic processes $\widetilde{W}_1$ *and* $\widetilde{W}_2$ *with decomposition given by*

$$\widetilde{W}_1(t) = W_1(t) - \int_0^t u(s)ds,$$

and

$$\widetilde{W}_2(t) = W_2(t) - \int_0^t \lambda(s)ds,$$

are $(\mathbb{F}, \mathbb{Q})$ *Brownian motions.*

The Robust Control Problem

$$
\begin{aligned}
&\sup_{(\pi,b)\in\mathcal{A}^{\mathbb{F}}} \inf_{\mathbb{Q}\in\mathcal{Q}} J(t,y) \\
&= \sup_{(\pi,b)\in\mathcal{A}^{\mathbb{F}}} \inf_{(u,\lambda)\in\mathcal{Y}} \mathbb{E}_{\mathbb{Q}}\left[U(Y(T)) + \frac{1}{2\beta}\int_t^T \left(u^2(s)+\lambda^2(s)\right) ds \right],
\end{aligned} \tag{6}
$$

subject to the state dynamics

$$
\begin{aligned}
dY(s) = \Big[& r + (\mu - r - \sigma k_2)\pi(s) + (\theta - k_3)\alpha b(s) + (\sigma\pi(s) - k_2)u(s) \\
& + (\alpha b(s) - k_3)\lambda(s) - \left(\mu_L(r) - k_2^2 - k_3^2\right) \Big] Y(s)ds + qds \\
& + (\sigma\pi(s) - k_2)Y(s)d\widetilde{W_1}(s) + (\alpha b(s) - k_3)Y(s)d\widetilde{W_2}(s),
\end{aligned}
$$

with initial condition $Y(s) = y > 0$.

The Robust Control Problem - Parameter β

- Departures from the reference probability model are penalized.
- These penalizations are weighted by the term $1/\beta$.
- $\beta > 0$ is referred to as the preference for robustness parameter, and serves as a measure to quantify the preference for robustness.
- Connection with constraint control!

Two interesting limiting cases:

- $\beta \to 0$: In this case, the fund manager fully trusts the model he/she is offered and seeks no robustness.
- $\beta \to \infty$: In this case, the fund manager has no faith in the model he/she faces and seeks alternative models with larger entropy.

On the solvability of the BI

- The value of the problem (in the Nash sense, if it exists) is defined as the solution of a second-order highly nonlinear PDE, known as the Bellman-Isaacs (BI) equation.
- Its solution is **crucial**; Optimal controls are defined as functions of the derivatives of this solution.

Is it possible to find a (smooth) solution to the HJBI ?

NOT IN GENERAL !!

There are three ways to proceed:

1. Guess a solution and pray !
2. Numerical Approximation.
3. Weak solutions (viscosity, mild, etc.).

Stochastic Differential Games

- The evolution of the underlying system is described by a Stochastic Differential Equation.
- The system is controlled by two (or more) players with conflicting goals.
- The controllers decide about their control process so as to drive the system to a desired state.

A robust control problem is written as a SDG:

- Player I. *Decision maker*: Chooses the control process.
- Player II. *Nature*: Chooses the model (probability measure)

Example of a Stochastic Differential Game

Figure: World Chess Championship 2016

Theorem (GENERAL SOLUTION)

Suppose that the fund manager has preference for robustness as described by the positive constant β. The optimal robust strategy is to invest in the stock, proportion of the fund's wealth equal to

$$\pi^*(t,y) = -\left(\frac{\mu - r}{\sigma} - k_2\right) \frac{V_y}{\sigma y \left(V_{yy} - \beta V_y^2\right)} + \frac{k_2}{\sigma}, \tag{7}$$

and in the zero coupon bond, proportion of the fund's wealth equal to

$$b^*(t,y) = -(\theta - k_3) \frac{V_y}{\alpha y \left(V_{yy} - \beta V_y^2\right)} + \frac{k_3}{\alpha}. \tag{8}$$

On the other hand, Nature chooses the worst-case scenario defined by

$$u^*(t,y) = \left(\frac{\mu - r}{\sigma} - k_2\right) \frac{\beta V_y^2}{V_{yy} - \beta V_y^2} \quad \text{and} \quad \lambda^*(t,y) = (\theta - k_3) \frac{\beta V_y^2}{V_{yy} - \beta V_y^2}. \tag{9}$$

In this case, the optimal robust value function is a smooth solution of the following nonlinear, second-order partial differential equation:

$$V_t + (\xi y + q) V_y - \frac{1}{2}\left[\left(\frac{\mu - r}{\sigma} - k_2\right)^2 + (\theta - k_3)^2\right] \frac{V_y^2}{V_{yy} - \beta V_y^2} = 0, \tag{10}$$

with $\xi = r - \mu_L + \frac{\mu - r}{\sigma} k_2 + \theta k_3$ and boundary condition $V(T,y) = U(y)$, assuming that such a solution exists and it satisfies the conditions $V_y > 0$ and $V_{yy} < 0$.

Theorem (Exponential Utility)

Assume Exponential utility $(U(y) = -\frac{1}{\gamma}e^{-\gamma y})$. *The optimal robust value function admits the form:*

$$V(t,x) = -\frac{1}{\gamma}\exp\left[-\gamma y e^{\delta(T-t)} + g(t)\right],\ \delta = \left(-k_1 + \theta k_3 + \frac{\mu - r}{\sigma}k_2\right), \tag{11}$$

where

$$g(t) = \frac{\gamma q}{\delta}\left(1 - e^{\delta(T-t)}\right) - \frac{\gamma(T-t)}{2(\beta+\gamma)}\left[\left(\frac{\mu-r}{\sigma} - k_2\right)^2 + (\theta - k_3)^2\right]. \tag{12}$$

In this case, the optimal robust strategy for the fund manager is to invest in the stock, proportion of the fund's wealth equal to

$$\pi^*(t,y) = \left(\frac{\mu-r}{\sigma} - k_2\right)\frac{e^{-\delta(T-t)}}{\sigma y(\beta+\gamma)} + \frac{k_2}{\sigma}, \tag{13}$$

furthermore, to invest in the zero coupon bond, proportion of the fund's wealth equal to

$$b^*(t,y) = (\theta - k_3)\frac{e^{-\delta(T-t)}}{\alpha y(\beta+\gamma)} + \frac{k_3}{\alpha}. \tag{14}$$

On the other hand, Nature chooses the worst-case scenario defined by

$$u^*(t) = -\left(\frac{\mu-r}{\sigma} - k_2\right)\frac{\beta}{\beta+\gamma} \ \ \textit{and}\ \ \lambda^*(t) = -(\theta - k_3)\frac{\beta}{\beta+\gamma}. \tag{15}$$

Numerical study of the optimal investment strategy

$$\boxed{\text{Euler-Maruyama scheme}} \longrightarrow \boxed{\text{Monte-Carlo approach}}$$

E-M: For a time step of size $\Delta t = T/N$ with $N = 2^{12}$ points, we define the step size in the Euler-Maruyama scheme as $\delta t = \Delta t$.

M-C: Simulate a large number M of paths of π^* and b^* in the time interval $[0, T]$ and at each time point we plot the average of M different values. We also use for each path a large number of points.

In what follows, unless stated otherwise, we let $M = 10000$, $T = 20$ years, $Y(0) = 1$, $q = 2\%$, $\gamma = 1.5$ and $\beta = 0.1$. The parameters of the financial market are chosen as $\mu = 10\%$, $r = 6\%$, $\sigma = 32\%$, $\theta = 15\%$, $\alpha = 30\%$, $k_1 = 5\%$ and $k_2 = k_3 = 8\%$.

Effect of Robustness on Optimal Strategies

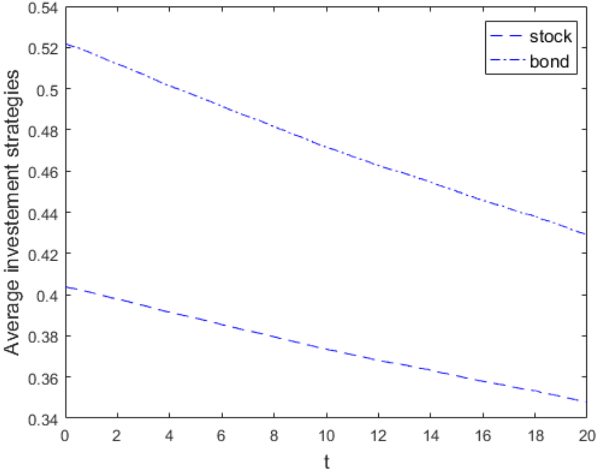


Figure: Average path of 10000 optimal investment strategy paths (bond and stock) in the case of the exponential utility function. **Here we let** $\beta = 0.1$.

Effect of Robustness on Optimal Strategies

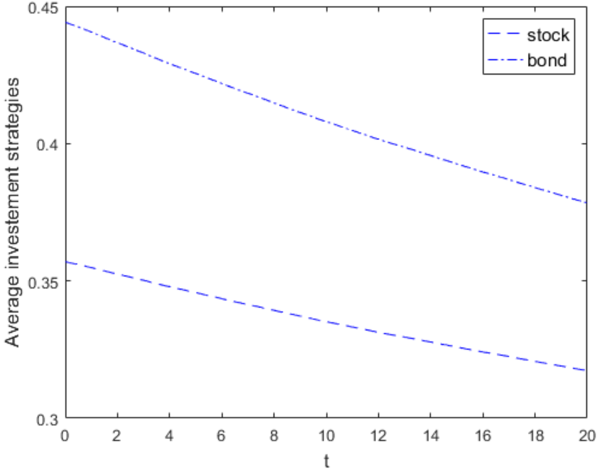


Figure: Average path of 10000 optimal investment strategy paths (bond and stock) in the case of the exponential utility function. **Here we let** $\beta = 0.8$.

Effect of Robustness on Optimal Strategies

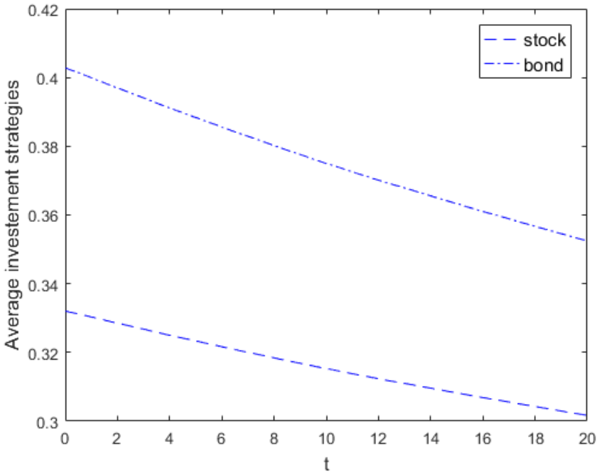


Figure: Average path of 10000 optimal investment strategy paths (bond and stock) in the case of the exponential utility function. **Here we let** $\beta = 1.5$.

Effect of Robustness on Optimal Strategies

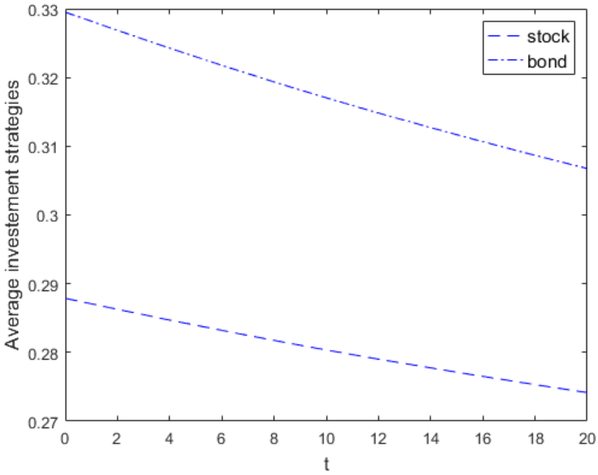


Figure: Average path of 10000 optimal investment strategy paths (bond and stock) in the case of the exponential utility function. **Here we let** $\beta = 3$.

Effect of Robustness on Optimal Strategies

We observe:

- As the level of the preference for robustness parameter β increases, the fund manager is expected to turn his/her attention in the bank account.
- This seems natural.
- The more the preference for robustness, the less the faith that in the reference model.
- In this case, the fund manager seeks to change to other models with larger relative entropy.

Effect of Initial Wealth Level on Optimal Strategies

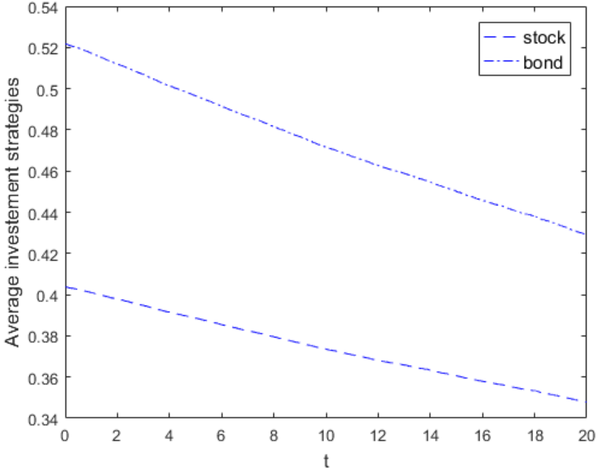


Figure: Average path of 10000 optimal investment strategy paths (bond and stock) in the case of the exponential utility function. **Here we let** $y_0 = 1$.

Effect of Initial Wealth Level on Optimal Strategies

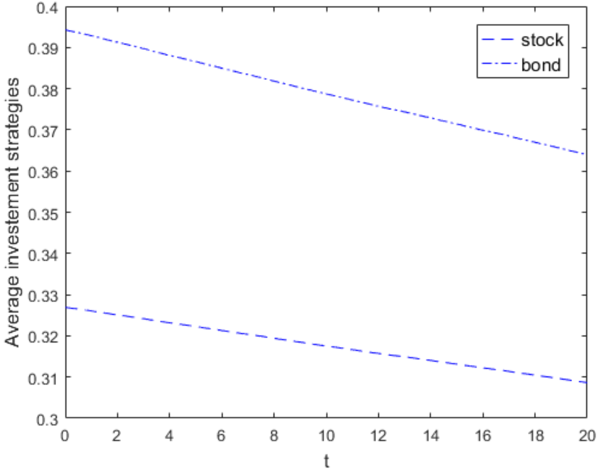


Figure: Average path of 10000 optimal investment strategy paths (bond and stock) in the case of the exponential utility function. **Here we let** $y_0 = 2$.

Effect of Initial Wealth Level on Optimal Strategies

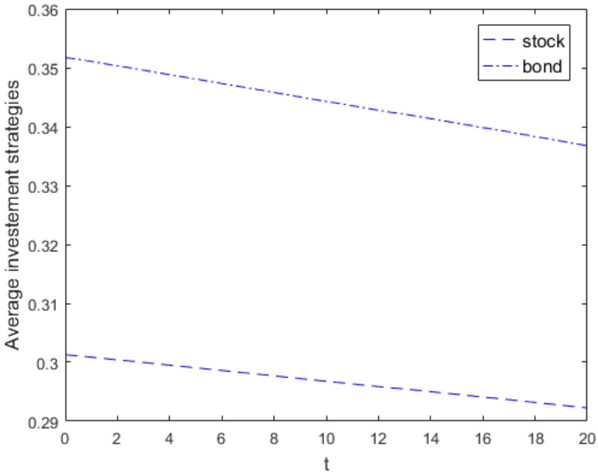


Figure: Average path of 10000 optimal investment strategy paths (bond and stock) in the case of the exponential utility function. **Here we let** $y_0 = 3$.

Effect of Initial Wealth Level on Optimal Strategies

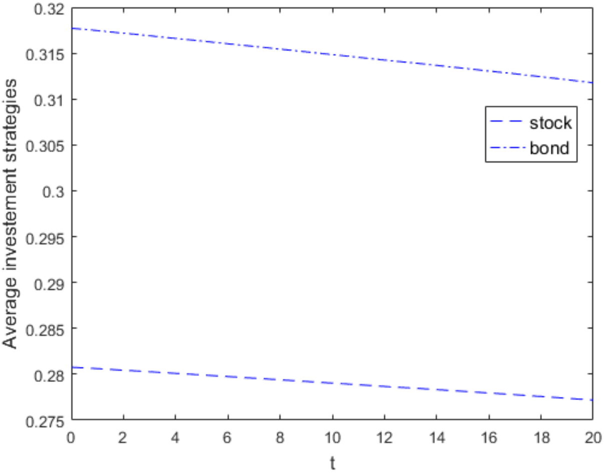


Figure: Average path of 10000 optimal investment strategy paths (bond and stock) in the case of the exponential utility function. **Here welet** $y_0 = 4$.

Effect of Initial Wealth Level on Optimal Strategies

We observe:

- As the initial relative wealth level increases, the fund manager is expected to turn his/her attention in the risk-free interest rate.
- This behavior is a direct consequence of the exponential utility function.
- As the fund's relative wealth increases, the fund manager (under an exponential attitude towards risk) decides to reduce the portfolio's exposure to the variability of the financial markets and increase its preference to the risk-free interest offered by the bank.

Conclusions - Outline

- The problem of providing supplementary pensions to the retirees has attracted a lot of attention, especially after the financial crisis.
- A popular solution to this problem is provided by pension fund schemes.
- According to the DC plan, every member of the fund contributes a fixed proportion of his/her income (before retirement), which are collected in an individual investment account
- the benefits to be received (after retirement) consist of a fraction of the true fund value and thus solely depend on the investment performance of the fund portfolio during its lifetime.
- The problem of optimal management of DC pension funds resorts to the problem of optimal portfolio selection.
- Techniques of Optimal Control are needed!

Conclusions - Outline

- In the present work, we study the problem of optimal management of DC pension funds under model uncertainty.
- Not the first attempt but our work is different in many aspects (e.g., we provide a detailed limited study in the paper - to be submitted for publication).
- Model is simplistic but can be extended to more realistic environments.
- Possible extensions:
  - Introduce more sources of stochasticity.
  - Calibrate results to real data.
  - Discrete time?
  - Modeling and solution under infinite-time Horizon and under Regimes?
- This work is the first step of a vast and dynamic research agenda.

References - Classical on Robust Control Theory

1. E. Anderson, L. Hansen and T. Sargent (2003). A quartet of semigroups for model specification, robustness, prices of risk, and model detection, Journal of the European Economic Association, **1**, 68-123.
2. E. Anderson, E. Ghysels and J. Juergens (2009). The impact of risk and uncertainty on expected returns, Journal of Financial Economics, **94**, 233-263.
3. L. Hansen and T. Sargent (2001). Robust control and model uncertainty, American Economic Review, **91**, 60-66. 317-337.

References - Applications of Robust Control Solutions in Closed Form

4. I. Baltas, A. Xepapadeas and A.N. Yannacopoulos (2018). Robust portfolio decisions for finanial institutions. Journal of Dynamics and Games, **5**, 61-94.
5. I. Baltas, A. Xepapadeas and A.N. Yannacopoulos, Robust control of parabolic stochastic partial differential equations under model uncertainty (2019). European Journal of Control, **46**, 1-13.
6. P. Maenhout (2005). Robust portfolio rules and asset pricing, The Review of Financial Studies, **17**, 951-983.
7. S. Mataramvura and B. Oksendal (2008). Risk minimizing portfolios and HJBI equations for stochastic differential games, Stochastics: An International Journal of Probability and Stochastic Processes, **80**, 317-337.

Thank you very much for your attention !

Information about the general project (leader):

- Marek Szczepánski: marek.szczepanski@put.poznan.pl

Information about the modeling part:

- Gerhard-Wilhelm Weber: gerhard.weber@put.poznan.pl
- Yiannis Baltas: jmpaltas@aegean.gr
- Athanasios N. Yannacopoulos:ayannaco@aueb.gr